

PAIRWISE k -SEMI-STRATIFIABLE BISPACES AND TOPOLOGICAL ORDERED SPACES

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ABSTRACT. In this paper, we continue to study pairwise (k -semi-)stratifiable bitopological spaces. Some new characterizations of pairwise k -semi-stratifiable bitopological spaces are provided. Relationships between pairwise stratifiable and pairwise k -semi-stratifiable bitopological spaces are further investigated, and an open question recently posed by Li and Lin in [18] is completely solved. We also study the quasi-pseudo-metrizability of a topological ordered space (X, τ, \preceq) . It is shown that if (X, τ, \preceq) is a ball transitive topological ordered C - and I -space such that τ is metrizable, then its associated bitopological space (X, τ^b, τ^d) is quasi-pseudo-metrizable. This result provides a partial affirmative answer to a problem in [15].

1. INTRODUCTION

Undoubtedly, topology and order are not only important topics in mathematics but also applicable in many other disciplines. For example, nonsymmetric notions of distance are needed for mathematical modelling in the natural, physical and cybernetic sciences and the corresponding topological notion is that of a quasi-metric or a quasi-pseudo-metric. The study of quasi-metrizable spaces naturally leads to the concepts of quasi-uniformities and bitopological spaces. In this aspect, Kelly's seminal paper [14] made pioneer contributions. On the other hand, the notions of a sober space and the Scott topology, in align with the investigation of partial orders and pre-orders, are useful in theoretical computer science in the study of algorithms which act on other algorithms. Moreover, finite topological spaces (i.e., finite pre-orders) can be used to construct a mathematical model of a video monitor screen which may be useful in computer graphics.

Since Kelly's work in [14], bitopological spaces have attracted the attention of many researchers. For example, Reilly [29] explored separation axioms for bitopological spaces, Cooke and Reilly [4] discussed the relationships between six definitions of bitopological compactness appeared in the literature, Raghavan and Reilly [28] introduced a notion of bitopological paracompactness and established

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a bitopological version of Michael's classical characterization of regular paracompact spaces. In addition to separation and covering properties, generalized metric properties have also been considered in the setting of bitopological spaces. In this direction, Fox [7] discussed the quasi-metrizability of bitopological spaces, pairwise stratifiable bitopological spaces and their generalizations have been introduced and studied by [12], [21] and [22].

Interplay between topology and order has been a very interesting area. In 1965, Nachbin's book [26] was published. This book is one of general references on the subject available today, and it covers results obtained by the author in his research on spaces with structures of order and topology. Among many topics in this line, McCartan [23] studied bicontinuous (herein called C -space and I -space) pre-ordered topological spaces and investigated the relationships between the topology of such a space and two associated convex topologies, and Faber [6] studied metrizability in generalized ordered spaces. Recently, there have been some renewed interests in the study of generalized metric properties in bitopological spaces and topological ordered spaces. Künzi and Mushaandja investigated the quasi-pseudo-metrizability of a topological ordered space in [15] and [25], respectively. They obtained some results related to the upper topology τ^{\natural} and the lower topology τ^b of a metrizable ordered space (X, τ, \preceq) which is both a C - and an I -space in the sense of Priestley [27]. Moreover, Li [16] as well as Li and Lin [17] carried on the study of pairwise (semi-)stratifiable bispaces and established some new characterizations for these classes of bitopological spaces. In a very recent paper [18], Li and Lin further introduced and studied the class of k -semi-stratifiable bitopological spaces.

The main purpose of this paper is to continue the study of pairwise k -semi-stratifiable bitopological spaces and their relationships with and applications to the quasi-pseudo-metrizability of a topological ordered spaces. For the sake of self-completeness, in Section 2, we introduce the necessary definitions and terminologies. In Section 3, we provide some new characterizations of pairwise k -semi-stratifiable bitopological spaces in terms of g -functions σ -cushioned pair k -networks and cs -networks. In Section 4, we consider some conditions under which a pairwise k -semi-stratifiable bitopological space is pairwise stratifiable. An open question posed in [18] is completely solved and results in [2] and [3] are extended to the setting of bitopological spaces. In the last section, we consider the quasi-pseudo-metrizability of a topological ordered spaces and provide a partial affirmative answer to an open problem of Künzi and Mushaandja in [15].

Our notations in this paper are standard. For any undefined concepts and terminologies, we refer the reader to [5] or [10].

2. PRELIMINARIES AND NOTATIONS

A *quasi-pseudo-metric* d on a nonempty set X is a non-negative real-valued function $d : X \times X \rightarrow \mathbb{R}_+$ such that (i) $d(x, x) = 0$ and (ii) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$. If d is a quasi-pseudo-metric on X , then the ordered pair (X, d) is called a *quasi-pseudo-metric space*. Every quasi-pseudo-metric d on X induces a topology $\tau(d)$ on X which has as a base the family $\{B_d(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. Every quasi-pseudo-metric d on X induces a conjugate quasi-pseudo-metric d^{-1} on X , defined by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$. A *bitopological space* [14] (for short, *bispace* [12]) is a triple (X, τ_1, τ_2) ,

where X is a nonempty set, topologies τ_1 and τ_2 are two topologies on X . A bisppace (X, τ_1, τ_2) is called *quasi-pseudo-metrizable*, if there is a quasi-pseudo-metric d on X such that $\tau(d) = \tau_1$ and $\tau(d^{-1}) = \tau_2$.

Let (X, τ_1, τ_2) be a bisppace. For $i = 1, 2$, let $\mathcal{F}_i(X)$ denote the family of all τ_i -closed subsets of X . For a subset A of X , let \overline{A}^{τ_i} and $\text{int}_{\tau_i}(A)$ denote the closure and interior of A with respect to τ_i , $i = 1, 2$, respectively. For $i, j = 1, 2$ with $i \neq j$, (X, τ_1, τ_2) is called *τ_i -semi-stratifiable with respect to τ_j* if there exists an operator $G_{ij} : \mathbb{N} \times \mathcal{F}_i(X) \rightarrow \tau_j$ satisfying (i) $H = \bigcap_{n \in \mathbb{N}} G_{ij}(n, H)$ for all $H \in \mathcal{F}_i(X)$, (ii) if $H, K \in \mathcal{F}_i(X)$ with $H \subseteq K$, then $G_{ij}(n, H) \subseteq G_{ij}(n, K)$ for all $n \in \mathbb{N}$. Furthermore, if G_{ij} satisfies (ii) and (i)' $H = \bigcap_{n \in \mathbb{N}} \overline{G_{ij}(n, H)}^{\tau_i}$ for all $H \in \mathcal{F}_i(X)$, then (X, τ_1, τ_2) is called *τ_i -stratifiable with respect to τ_j* . Moreover, (X, τ_1, τ_2) is called *pairwise (semi-)stratifiable* [7], [12] and [22], if it is both τ_1 -semi-stratifiable with respect to τ_2 and τ_2 -semi-stratifiable with respect to τ_1 .

Recently, Li and Lin [18] introduced the concept of a pairwise k -semi-stratifiable bisppace, which is a natural extension of a k -semi-stratifiable space introduced in [19] to the setting of bispaces. For $i, j = 1, 2$ with $i \neq j$, a bisppace (X, τ_1, τ_2) is called *τ_i - k -semi-stratifiable with respect to τ_j* if there exists an operator $G_{ij} : \mathbb{N} \times \mathcal{F}_i(X) \rightarrow \tau_j$ satisfying (i) $H = \bigcap_{n \in \mathbb{N}} G_{ij}(n, H)$ for all $H \in \mathcal{F}_i(X)$, (ii) if $H, K \in \mathcal{F}_i(X)$ with $H \subseteq K$, then $G_{ij}(n, H) \subseteq G_{ij}(n, K)$ for all $n \in \mathbb{N}$, (iii) if $K \subseteq X$ is τ_i -compact and $H \in \mathcal{F}_i(X)$ such that $H \cap K = \emptyset$, then $K \cap G_{ij}(n, H) = \emptyset$ for some $n \in \mathbb{N}$. In addition, (X, τ_1, τ_2) is called *pairwise k -semi-stratifiable* [18] if it is both τ_1 - k -semi-stratifiable with respect to τ_2 and τ_2 - k -semi-stratifiable with respect to τ_1 .

The next lemma, which gives an important dual characterization of pairwise k -semi-stratifiable bispaces, will be used in the sequel.

Lemma 2.1 ([18]). *A bisppace (X, τ_1, τ_2) is pairwise k -semi-stratifiable if, and only if, for any $i, j = 1, 2$ with $i \neq j$, there is an operator $F_{ij} : \mathbb{N} \times \tau_i \rightarrow \mathcal{F}_j(X)$ satisfying*

- (1) $U = \bigcup_{n \in \mathbb{N}} F_{ij}(n, U)$ for all $U \in \tau_i$;
- (2) if $U, V \in \tau_i$ with $U \subseteq V$, then $F_{ij}(n, U) \subseteq F_{ij}(n, V)$ for all $n \in \mathbb{N}$;
- (3) if $K \subseteq X$ is τ_i -compact and $U \in \tau_i$ with $K \subseteq U$, then $K \subseteq F_{ij}(n, U)$ for some $n \in \mathbb{N}$.

In addition, the operator F_{ij} can be required to be monotone with respect to n , that is, $F_{ij}(n, U) \subseteq F_{ij}(n+1, U)$ for all $n \in \mathbb{N}$ and all $U \in \tau_i$.

By definition, every pairwise stratifiable bisppace is pairwise k -semi-stratifiable, and every pairwise k -semi-stratifiable bisppace is pairwise semi-stratifiable. Recall that a bisppace (X, τ_1, τ_2) is said to be *pairwise monotonically normal* [21] if to each pair (H, K) of disjoint subsets of X such that $H \in \mathcal{F}_i(X)$ and $K \in \mathcal{F}_j(X)$ ($i, j = 1, 2$ and $i \neq j$), we can assign a set $D_{ij}(H, K) \in \tau_j$ such that (i)

$$H \subseteq D_{ij}(H, K) \subseteq \overline{D_{ij}(H, K)}^{\tau_i} \subseteq X \setminus K,$$

(ii) if the pairs (H, K) and (H', K') satisfy $H \subseteq H'$ and $K' \subseteq K$, then $D_{ij}(H, K) \subseteq D_{ij}(H', K')$.

The following result, established by Marín and Romaguera in [21], extends the celebrated result of Heath et al. in [13] on monotonically normal spaces.

Theorem 2.2 ([21]). *A bisppace (X, τ_1, τ_2) is pairwise stratifiable if, and only if, it is a pairwise monotonically normal and pairwise semi-stratifiable bisppace.*

Corollary 2.3. *A pairwise monotonically normal and pairwise k semi-stratifiable bispaces is pairwise stratifiable.*

A topological ordered space (X, τ, \preceq) is a nonempty set X endowed with a topology τ and a partial order \preceq . A subset A of X is said to be an *upper set* of X if $x \preceq y$ and $x \in A$ imply that $y \in A$. Similarly, we say that a subset A of X is a *lower set* of X if $y \preceq x$ and $x \in A$ imply that $y \in A$. Let τ^\flat denote the collection of τ -open lower sets of X and τ^\sharp denote the collection of τ -open upper sets of X . Then, τ^\flat and τ^\sharp are two topologies on X and thus $(X, \tau^\flat, \tau^\sharp)$ is a bispaces. For any subset A of X , $i(A)$ (resp. $d(A)$) will denote the intersection of all upper (lower) sets of X containing A . Note that $i(A)$ (resp. $d(A)$) is the smallest upper (resp. lower) set containing A . It is easy to see that $A = i(A)$ if, and only if, A is an upper set. Similarly, $A = d(A)$ if, and only if, A is a lower set. Following Priestley [27], we recall that a topological ordered space (X, τ, \preceq) is said to be a *C-space* if $d(F)$ and $i(F)$ are closed whenever F is a closed subset of X . Similarly, a topological ordered space (X, τ, \preceq) is called an *I-space* if $d(G)$ and $i(G)$ are open whenever G is an open subset of X .

3. SOME NEW CHARACTERIZATIONS OF PAIRWISE k -SEMI-STRATIFIABLE BISPACES

In [18], Li and Lin characterized pairwise k -semi-stratifiable bispaces in terms of pairwise g -functions and extensions of semi-continuous functions. In this section, we continue to investigate how to characterize pairwise k -semi-stratifiable bispaces. Our first two results can be regarded as either improvements or extensions of a theorem in [18]. In addition, we also use cushioned pair k -networks and *cs*-networks to characterize pairwise k -semi-stratifiable bispaces.

Let (X, τ_1, τ_2) be a bispaces. A *pairwise g -function* on (X, τ_1, τ_2) is a pair of functions (g_1, g_2) such that for $i = 1, 2$, $g_i : \mathbb{N} \times X \rightarrow \tau_i$ satisfies $x \in g_i(n, x)$ and $g_i(n+1, x) \subseteq g_i(n, x)$ for all $n \in \mathbb{N}$ and $x \in X$. A pairwise family $\mathcal{B} = \{(B_\alpha^1, B_\alpha^2) : \alpha \in \Delta\}$ of subsets of X is called *τ_j -cushioned*, where $j = 1, 2$, if for any $\Delta' \subseteq \Delta$,

$$\overline{\bigcup \{B_\alpha^1 : \alpha \in \Delta'\}}^{\tau_j} \subseteq \bigcup \{B_\alpha^2 : \alpha \in \Delta'\}.$$

Furthermore, if \mathcal{B} is a countable union of τ_j -cushioned families, then it is called *σ - τ_j -cushioned*. A pairwise family $\mathcal{B} = \{(B_\alpha^1, B_\alpha^2) : \alpha \in \Delta\}$ is called a *pair τ_i - k -network* if for any τ_i -compact set K and any set $U \in \tau_i$ with $K \subseteq U$, there is a finite subset Δ' of Δ such that

$$K \subseteq \bigcup \{B_\alpha^1 : \alpha \in \Delta'\} \subseteq \bigcup \{B_\alpha^2 : \alpha \in \Delta\} \subseteq U.$$

Our first result improves the equivalence of (1) and (2) in [18, Theorem 2.1].

Theorem 3.1. *Let (X, τ_1, τ_2) be a bispaces such that (X, τ_i) is T_1 -space for $i = 1, 2$. Then (X, τ_1, τ_2) is pairwise k -semi-stratifiable if, and only if, there is a pairwise g -function (g_1, g_2) such that for $i, j = 1, 2$ with $i \neq j$, if K is a τ_i -compact set and H is a τ_i -closed set with $K \cap H = \emptyset$, then*

$$K \cap \left(\bigcup \{g_j(m, x) : x \in H\} \right) = \emptyset$$

for some $m \in \mathbb{N}$.

Proof. Necessity. Suppose that (X, τ_1, τ_2) is pairwise k -semi-stratifiable. For $i, j = 1, 2$ and $i \neq j$, let $G_{ij} : \mathbb{N} \times \mathcal{F}_i(X) \rightarrow \tau_j$ be an operator satisfying the definition of a pairwise k -semi-stratifiable bispaces. Without loss of generality, G_{ij} can be required to be monotone with respect to n . Define a function $g_j : \mathbb{N} \times X \rightarrow \tau_j$ such that $g_j(n, x) = G_{ij}(n, \{x\})$ for all $n \in \mathbb{N}$ and $x \in X$. Clearly, (g_1, g_2) is a pairwise g -function. If K is a τ_i -compact subset and H is a τ_i -closed subset with $K \cap H = \emptyset$, then $K \cap G_{ij}(m, H) = \emptyset$ for some $m \in \mathbb{N}$. Note that

$$\bigcup \{g_j(m, x) : x \in H\} \subseteq G_{ij}(m, H).$$

It follows that

$$K \cap \left(\bigcup \{g_j(m, x) : x \in H\} \right) = \emptyset.$$

Sufficiency. Let (g_1, g_2) be a pairwise g -function satisfying the assumption in the theorem. For each τ_i -closed subset H and $n \in \mathbb{N}$, define

$$G_{ij}(n, H) = \bigcup \{g_j(n, x) : x \in H\}.$$

We shall verify that G_{ij} is an operator satisfying conditions (i) in the definition of a pairwise k -semi-stratifiable bitopological space, as (ii) and (iii) hold trivially. It is clear that $H \subseteq \bigcap_{n \in \mathbb{N}} G_{ij}(n, H)$. If $p \notin H$, as $\{p\}$ is compact, then the assumption in the theorem implies that there must be some $m \in \mathbb{N}$ such that $p \notin \bigcup \{g_j(m, x) : x \in H\}$. It follows that $p \notin G_{ij}(m, H)$. Thus, $H = \bigcap_{n \in \mathbb{N}} G_{ij}(n, H)$. \square

Theorem 3.2. *Let (X, τ_1, τ_2) be a bispaces such that (X, τ_i) is Hausdorff for $i = 1, 2$. Then the following statements are equivalent.*

- (1) (X, τ_1, τ_2) is pairwise k -semi-stratifiable.
- (2) There is a pairwise g -function (g_1, g_2) such that for $i, j = 1, 2$ with $i \neq j$, if $\{x_n : n \in \mathbb{N}\}$ is a sequence τ_i -convergent to p and H is a τ_i -closed subset with $(\{p\} \cup \{x_n : n \in \mathbb{N}\}) \cap H = \emptyset$, then

$$(\{p\} \cup \{x_n : n \in \mathbb{N}\}) \cap \left(\bigcup \{g_j(m, x) : x \in H\} \right) = \emptyset$$

for some $m \in \mathbb{N}$.

- (3) There is a pairwise g -function (g_1, g_2) such that for $i, j = 1, 2$ with $i \neq j$, if $\{x_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$ are two sequences in X with $\{x_n : n \in \mathbb{N}\}$ τ_i -convergent to p and $x_n \in g_j(n, y_n)$ for all $n \in \mathbb{N}$, then $\{y_n : n \in \mathbb{N}\}$ is τ_i -convergent to p .

Proof. (1) \Rightarrow (2) follows directly from Theorem 3.1, as $\{p\} \cup \{x_n : n \in \mathbb{N}\}$ is compact.

(2) \Rightarrow (3). Let (g_1, g_2) be a pairwise g -function satisfying (2). Let $\{x_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$ be two sequences in X such that $\{x_n : n \in \mathbb{N}\}$ is τ_i -convergent to p and $x_n \in g_j(n, y_n)$. Assume that $\{y_n : n \in \mathbb{N}\}$ is not τ_i -convergent to p . Then $\{y_n : n \in \mathbb{N}\}$ has a subsequence $\{y_{n_k} : k \in \mathbb{N}\}$ such that $p \notin \overline{\{y_{n_k} : k \in \mathbb{N}\}}^{\tau_i}$. Put $H = \overline{\{y_{n_k} : k \in \mathbb{N}\}}^{\tau_i}$. Since $\{x_{n_k} : k \in \mathbb{N}\}$ is τ_i -convergent to p , then we can assume that $x_{n_k} \notin H$ for all $k \geq 1$. Thus, by (2), there must be an $m \in \mathbb{N}$ such that

$$(\{p\} \cup \{x_{n_k} : k \in \mathbb{N}\}) \cap \left(\bigcup \{g_j(m, x) : x \in H\} \right) = \emptyset.$$

On the other hand,

$$x_{n_m} \in g_j(n_m, y_{n_m}) \subseteq g_j(m, y_{n_m}) \subseteq \bigcup \{g_j(m, x) : x \in H\}.$$

A contradiction occurs.

(3) \Rightarrow (1). A proof has been given in [18]. \square

In [8], k -semi-stratifiable spaces are defined in terms of σ -cushioned pair k -networks, which is different from (but equivalent to) that given in [19]. Our next result, which just confirms that the same thing holds in the setting of bispaces, provides characterizations of a pairwise k -semi-stratifiable bispace in terms of cushioned pair k -networks.

Theorem 3.3. *A bispace (X, τ_1, τ_2) is τ_1 - k -semi-stratifiable with respect to τ_2 if, and only if, it has a σ - τ_2 -cushioned pair τ_1 - k -network.*

Proof. Necessity. Let $F_{12} : \mathbb{N} \times \tau_1 \rightarrow \mathcal{F}_2(X)$ be an operator satisfying conditions (1)-(3) in Lemma 2.1 such that F_{12} is also monotone with respect to n . For each $n \in \mathbb{N}$, define $\Delta_n = \tau_1$ and

$$\mathcal{B}_n = \{(B_U^1, B_U^2) : B_U^1 = F_{12}(n, U), B_U^2 = U \text{ and } U \in \Delta_n\}.$$

We claim that \mathcal{B}_n is τ_2 -cushioned. Indeed, if $\mathcal{U} \subseteq \Delta_n$, by condition (2) in Lemma 2.1, $F_{12}(n, U) \subseteq F_{12}(n, \bigcup \mathcal{U})$ for any $U \in \mathcal{U}$. It follows that

$$\begin{aligned} \overline{\bigcup \{B_U^1 : U \in \mathcal{U}\}}^{\tau_2} &= \overline{\bigcup \{F_{12}(n, U) : U \in \mathcal{U}\}}^{\tau_2} \\ &\subseteq F_{12}\left(n, \bigcup \mathcal{U}\right) \\ &\subseteq \bigcup \{B_U^2 : U \in \mathcal{U}\}, \end{aligned}$$

which implies that each \mathcal{B}_n is τ_2 -cushioned. Thus, $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ is σ - τ_2 -cushioned. Let K be a τ_1 -compact subset of X and $U \in \tau_1$ with $K \subseteq U$. By condition (3) in Lemma 2.1, there must be some $m \in \mathbb{N}$ such that $K \subseteq F_{12}(m, U) \subseteq U$. Then $\Delta' = \{U\} \subseteq \Delta_m$ and

$$K \subseteq \bigcup \{B_U^1 : U \in \Delta'\} \subseteq \bigcup \{B_U^2 : U \in \Delta'\} \subseteq U,$$

which implies that \mathcal{B} is also a pair τ_1 - k -network.

Sufficiency. Let $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ be a σ - τ_2 -cushioned pair τ_1 - k -network, that is, \mathcal{B} is a pair τ_1 - k -network and for each $n \in \mathbb{N}$, $\mathcal{B}_n = \{(B_\alpha^1, B_\alpha^2) : \alpha \in \Delta_n\}$ is a τ_2 -cushioned family. Without loss of generality, for each $n \in \mathbb{N}$, we can assume that $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$ and \mathcal{B}_n is closed under finite union. Define $F_{12} : \mathbb{N} \times \tau_1 \rightarrow \mathcal{F}_2(X)$ such that for each $n \in \mathbb{N}$ and each $U \in \tau_1$,

$$F_{12}(n, U) := \overline{\bigcup \{B_\alpha^1 : B_\alpha^2 \subseteq U \text{ and } \alpha \in \Delta_n\}}^{\tau_2}.$$

First of all, as \mathcal{B}_n is τ_2 -cushioned, we have

$$F_{12}(n, U) \subseteq \bigcup \{B_\alpha^2 : B_\alpha^2 \subseteq U \text{ and } \alpha \in \Delta_n\} \subseteq U.$$

For each $x \in U$, as \mathcal{B} is a pair τ_1 - k -network and $\{x\}$ is compact, there exist an $m \in \mathbb{N}$ and an $\alpha \in \Delta_m$ such that $x \in B_\alpha^1 \subseteq B_\alpha^2 \subseteq U$. It follows that $x \in F_{12}(m, U)$, which implies that $\bigcup_{n \in \mathbb{N}} F_{12}(n, U) = U$. It is clear that if $U, V \in \tau_1$ with $U \subseteq V$, then $F_{12}(n, U) \subseteq F_{12}(n, V)$ for any $n \in \mathbb{N}$. Finally, if K is τ_1 -compact and $U \in \tau_1$ with $K \subseteq U$, similar to what have done previously, there exist an $m \in \mathbb{N}$ and an $\alpha \in \Delta_m$ such that $K \in B_\alpha^1 \subseteq B_\alpha^2 \subseteq U$. This implies that $K \subseteq F_{12}(m, U) \subseteq U$. Therefore, we have checked that F_{12} is an operator satisfying conditions (1)-(3) in Lemma 2.1. \square

Corollary 3.4. *A bispaces (X, τ_1, τ_2) is pairwise k -semi-stratifiable if, and only if, it has a σ - τ_j -cushioned pair τ_i - k -network for each pair of $i, j = 1, 2$ with $i \neq j$.*

Let (X, τ_1, τ_2) be a bispaces and $x \in X$ be a point. A family \mathcal{P}_x of subsets of X is called a τ_i -cs-network at x [11] if for every sequence $\{x_n : n \in \mathbb{N}\}$ that is τ_i -convergent to x and an arbitrary open neighborhood U of x in (X, τ_i) , there exist an $m \in \mathbb{N}$ and an element $P \in \mathcal{P}_x$ such that

$$\{x\} \cup \{x_n : n \geq m\} \subseteq P \subseteq U.$$

If each point x in X has a τ_i -cs-network \mathcal{P}_x , then $\bigcup_{x \in X} \mathcal{P}_x$ is called τ_i -cs-network for (X, τ_i) .

In [8], Gao characterized k -semi-stratifiable spaces in terms of cs-networks. At the end of this section, we establish a similar result in the setting of bispaces.

Theorem 3.5. *Let (X, τ_1, τ_2) be a bispaces such that (X, τ_i) is Hausdorff for $i = 1, 2$. Then (X, τ_1, τ_2) is pairwise k -semi-stratifiable if, and only if, for any $i, j = 1, 2$ with $i \neq j$, there is an operator $F_{ij} : \mathbb{N} \times \tau_i \rightarrow \mathcal{F}_j(X)$ satisfying*

- (1) $U = \bigcup_{n \in \mathbb{N}} F_{ij}(n, U)$ for all $U \in \tau_i$;
- (2) if $U, V \in \tau_i$ with $U \subseteq V$, then $F_{ij}(n, U) \subseteq F_{ij}(n, V)$ for all $n \in \mathbb{N}$;
- (3) for each $U \in \tau_i$, $\{F_{ij}(n, U) : n \in \mathbb{N}\}$ is a τ_i -cs-network at every point of U .

In addition, the operator F_{ij} can be required to be monotone with respect to n , that is, $F_{ij}(n, U) \subseteq F_{ij}(n+1, U)$ for all $n \in \mathbb{N}$ and all $U \in \tau_i$.

Proof. The necessity is trivial by Lemma 2.1, as $\{x\} \cup \{x_n : n \in \mathbb{N}\}$ is τ_i -compact for any sequence $\{x_n : n \in \mathbb{N}\}$ in X which is τ_i -convergent to a point $x \in X$.

Sufficiency. Suppose that for any $i, j = 1, 2$ with $i \neq j$, there is an operator $F_{ij} : \mathbb{N} \times \tau_i \rightarrow \mathcal{F}_j(X)$ satisfying conditions (1)–(3) above and monotonicity. We only need to verify condition (3) in Lemma 2.1. First, note that these conditions imply that each point x is a G_δ -set in both (X, τ_1) and (X, τ_2) . Suppose that there are a τ_i -compact set K and a $U \in \tau_i$ with $K \subseteq U$, but $K \not\subseteq F_{ij}(n, U)$ for any $n \in \mathbb{N}$. Then, there is a sequence $\{x_n : n \in \mathbb{N}\} \subseteq K$ such that $x_n \notin F_{ij}(n, U)$ for any $n \in \mathbb{N}$. Since K is τ_i -compact and points are G_δ in (X, τ_i) , then $\{x_n : n \in \mathbb{N}\}$ must have a subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ which is τ_i -convergent to a point $x \in K$ and $x_{n_k} \notin F_{ij}(n_k, U)$ for all $k \in \mathbb{N}$. By condition (3) above, there exist an $m_0 \in \mathbb{N}$ and an $n_0 \in \mathbb{N}$ such that

$$\{x\} \cup \{x_{n_k} : k \geq m_0\} \subseteq F_{ij}(n_0, U) \subseteq U.$$

It follows that for any $k \in \mathbb{N}$ with $k \geq m_0$ and $n_k \geq n_0$, we have

$$x_{n_k} \in F_{ij}(n_0, U) \subseteq F_{ij}(n_k, U).$$

Apparently, this contradicts with the choice of x_{n_k} . We have verified that condition (3) in Lemma 2.1 is satisfied, and thus (X, τ_1, τ_2) is pairwise k -semi-stratifiable. \square

Note that Theorems 3.3 and 3.5 may shed some light on relationships between pairwise k -semi-stratifiability and the other generalized metric properties of bispaces studied in [25] and other places.

4. WHEN IS A PAIRWISE k -SEMI-STRATIFIABLE BISPACES
PAIRWISE STRATIFIABLE?

In this section, we consider the problem when a pairwise k -semi-stratifiable bispaces is pairwise stratifiable. An open question posed in [18] is completely solved, and some results in [2], [3] and [8] are extended to the setting of bispaces.

Recall that a topological space (X, τ) is said to be *Fréchet*, if for every nonempty subset $A \subseteq X$ and every point $x \in \overline{A}^\tau$, there is a sequence $\{x_n : n \in \mathbb{N}\} \subseteq A$ such that $\{x_n : n \in \mathbb{N}\}$ converges to x .

In a recent paper [18], Li and Lin posed the following open question (see [18, Question 3.4]).

Question 4.1 ([18]). *Is a pairwise k -semi-stratifiable bispaces (X, τ_1, τ_2) pairwise stratifiable if (X, τ_i) is a Fréchet space for each $i = 1, 2$?*

Our next theorem answers Question 4.1 affirmatively. Note that our theorem also extends a result in [8] to the setting of bispaces.

Theorem 4.2. *Let (X, τ_1, τ_2) be a pairwise k -semi-stratifiable bispaces. If both (X, τ_1) and (X, τ_2) are Fréchet spaces, then (X, τ_1, τ_2) is pairwise stratifiable.*

Proof. In the light of Corollary 2.3, we need to show that (X, τ_1, τ_2) is pairwise monotonically normal. For any fixed $i, j = 1, 2$ with $i \neq j$, let $F_{ij} : \mathbb{N} \times \tau_i \rightarrow \mathcal{F}_j(X)$ be an operator that is monotone with respect to n and satisfies (1)-(3) in Lemma 2.1. For each pair (H, K) of disjoint subsets of X such that $H \in \mathcal{F}_i(X)$ and $K \in \mathcal{F}_j(X)$, define $D_{ij}(H, K)$ by

$$D_{ij}(H, K) := \text{int}_{\tau_j} \left(\bigcup_{n \in \mathbb{N}} (F_{ji}(n, X \setminus K) \setminus F_{ij}(n, X \setminus H)) \right)$$

Next, we shall verify that $D_{ij}(\cdot, \cdot)$ satisfies all conditions in the definition of a pairwise monotonically normal bispaces.

Clearly, $D_{ij}(H, K) \in \tau_j$ and $D_{ij}(\cdot, \cdot)$ satisfies condition (ii) in the definition of a pairwise monotonically normal bispaces. Also note that $D_{ij}(H, K) \subseteq X \setminus K$ holds trivially.

Claim 1. $H \subseteq D_{ij}(H, K)$.

Proof of Claim 1. Suppose that there is a point $x_0 \in H \setminus D_{ij}(H, K)$. Then

$$\begin{aligned} x_0 &\in X \setminus \text{int}_{\tau_j} \left(\bigcup_{n \in \mathbb{N}} (F_{ji}(n, X \setminus K) \setminus F_{ij}(n, X \setminus H)) \right) \\ &= \overline{X \setminus \bigcup_{n \in \mathbb{N}} (F_{ji}(n, X \setminus K) \setminus F_{ij}(n, X \setminus H))}^{\tau_j}. \end{aligned}$$

Since (X, τ_j) is a Fréchet space, there is a sequence $\{x_n : n \in \mathbb{N}\}$ such that

$$\{x_n : n \in \mathbb{N}\} \subseteq X \setminus \bigcup_{n \in \mathbb{N}} (F_{ji}(n, X \setminus K) \setminus F_{ij}(n, X \setminus H))$$

and $\{x_n : n \in \mathbb{N}\}$ is τ_j -convergent to x_0 . Note that $x_0 \in H$ implies that $x_0 \in X \setminus K$. Thus, there is an $m \in \mathbb{N}$ such that

$$C = \{x_0\} \cup \{x_n : n \geq m\} \subseteq X \setminus K.$$

By (3) in Lemma 2.1, there is an $m' \geq m$ such that $C \subseteq F_{ji}(m', X \setminus K)$. On the other hand, note that there must be some $p \geq m'$ such that $x_p \notin F_{ij}(m', X \setminus H)$. Otherwise, as $F_{ij}(m', X \setminus H)$ is τ_j -closed, we conclude that

$$x_0 \in F_{ij}(m', X \setminus H) \subseteq X \setminus H,$$

which contradicts with the fact $x_0 \in H$. It follows that

$$\begin{aligned} x_p &\in F_{ji}(m', X \setminus K) \setminus F_{ij}(m', X \setminus H) \\ &\subseteq \bigcup_{n \in \mathbb{N}} (F_{ji}(n, X \setminus K) \setminus F_{ij}(n, X \setminus H)). \end{aligned}$$

This certainly contradicts with the selection of $\{x_n : n \in \mathbb{N}\}$. Hence, Claim 1 has been verified. \square

Claim 2. $\overline{D_{ij}(H, K)}^{\tau_i} \subseteq X \setminus K$.

Proof of Claim 2. Suppose that there is a point $x_0 \in \overline{D_{ij}(H, K)}^{\tau_i} \cap K$. Since (X, τ_i) is a Fréchet space, there is a sequence $\{x_n : n \in \mathbb{N}\} \subseteq D_{ij}(H, K)$ such that $\{x_n : n \in \mathbb{N}\}$ is τ_i -convergent to x_0 . Note that $x_0 \in K$ implies $x_0 \in X \setminus H$. Thus, there exists an $m \in \mathbb{N}$ such that

$$\{x_0\} \cup \{x_n : n \geq m\} \subseteq X \setminus H.$$

By condition (3) in Lemma 2.1, there exists some $p \geq m$ such that

$$\{x_0\} \cup \{x_n : n \geq m\} \subseteq F_{ij}(p, X \setminus H).$$

By the selection of $\{x_n : n \in \mathbb{N}\}$, we know that

$$\{x_n : n \geq m\} \subseteq \bigcup_{n \in \mathbb{N}} (F_{ji}(n, X \setminus K) \setminus F_{ij}(n, X \setminus H)),$$

which implies that

$$\{x_n : n \geq m\} \subseteq \bigcup_{n=1}^{p-1} (F_{ji}(n, X \setminus K) \setminus F_{ij}(n, X \setminus H)).$$

It follows that there are a $q \in \mathbb{N}$ with $1 \leq q < p$ and a subsequence $\{x_{n_k} : k \geq m\}$ of $\{x_n : n \in \mathbb{N}\}$ such that

$$\{x_{n_k} : k \geq m\} \subseteq F_{ji}(q, X \setminus K) \setminus F_{ij}(q, X \setminus H).$$

We conclude that $x_0 \in F_{ji}(q, X \setminus K) \subseteq X \setminus K$. This contradicts with the selection of x_0 , and thus Claim 2 has been verified. \square

Combining Claims 1 and 2, we see that $D_{ij}(\cdot, \cdot)$ also satisfies condition (i) in the definition of a pairwise monotonically normal bspace. \square

Let (X, τ) be a topological space, and let $x \in X$ be a point. The collection of neighborhoods of x in (X, τ) is denoted by $\mathcal{N}(\tau, x)$. We shall consider the following $\mathcal{G}(\tau, x)$ -game played in (X, τ) between two players: α and β . Player α goes first and chooses a point $x_1 \in X$. Player β then responds by choosing $U_1 \in \mathcal{N}(\tau, x)$. Following this, α must select another (possibly the same) point $x_2 \in U_1$ and in turn β must again respond to this by choosing (possibly the same) $U_2 \in \mathcal{N}(\tau, x)$. The players repeat this procedure infinitely many times, and produce a play $(x_1, U_2, x_2, U_2, \dots, x_n, U_n, \dots)$ in the $\mathcal{G}(\tau, x)$ -game, satisfying $x_{n+1} \in U_n$ for all $n \in \mathbb{N}$. We shall say that β wins a play $(x_1, U_2, x_2, U_2, \dots, x_n, U_n, \dots)$ if the

sequence $\{x_n : n \in \mathbb{N}\}$ has a cluster point in X . Otherwise, α is said to *have won* the play. By a *strategy* s for β , we mean a ‘rule’ that specifies each move of β in every possible situation. More precisely, a strategy s for β is an $\mathcal{N}(\tau, x)$ -valued function. We shall call a finite sequence $\{x_1, x_2, \dots, x_n\}$ or an infinite sequence $\{x_1, x_2, \dots\}$ an *s-sequence* if $x_{i+1} \in s(x_1, x_2, \dots, x_i)$ for each i such that $1 \leq i < n$ or $x_{n+1} \in s(x_1, x_2, \dots, x_n)$ for each $n \in \mathbb{N}$. A strategy s for player β is called a *winning strategy* if each infinite *s-sequence* has a cluster point in X . Finally, we call x a $\mathcal{G}(\tau)$ -point if player β has a winning strategy for the $\mathcal{G}(\tau, x)$ -game. In addition, if every point of X is a $\mathcal{G}(\tau)$ -point, then (X, τ) is called a \mathcal{G} -space [1]. The notion of \mathcal{G} -spaces is a common generalization of the concepts of q -spaces in [24] and W -spaces in [9].

The next result extends [2, Theorem 2.2.8] and [3, Theorem 3.2] to the setting of bispaces.

Theorem 4.3. *Let (X, τ_1, τ_2) be a pairwise k -semi-stratifiable bispace. If both (X, τ_1) and (X, τ_2) are regular and \mathcal{G} -spaces, then (X, τ_1, τ_2) is pairwise stratifiable.*

Proof. Let (g_1, g_2) be a pairwise g -function as described in condition (3) of Theorem 3.2. For $i, j = 1, 2$ with $i \neq j$, we define $G_{ij} : \mathbb{N} \times \mathcal{F}_i(X) \rightarrow \tau_j$ such that for each $n \in \mathbb{N}$ and each $H \in \mathcal{F}_i(X)$,

$$G_{ij}(n, H) = \bigcup \{g_j(n, x) : x \in H\}.$$

Clearly, if $H, K \in \mathcal{F}_i(X)$ with $H \subseteq K$, then $G_{ij}(n, H) \subseteq G_{ij}(n, K)$ for all $n \in \mathbb{N}$. Furthermore, it is easy to see that $H \subseteq \bigcap_{n \in \mathbb{N}} \overline{G_{ij}(n, H)}^{\tau_i}$ for all $H \in \mathcal{F}_i(X)$.

Suppose that there are a point $x \in X$ and a τ_i -closed subset H in X with $x \notin H$, but $x \in \overline{G_{ij}(n, H)}^{\tau_i}$ for every $n \in \mathbb{N}$. First, we choose some τ_i -open neighborhood U of x such that $\overline{U}^{\tau_i} \cap H = \emptyset$. Since (X, τ_i) is a \mathcal{G} -space, β has a winning strategy s for the $\mathcal{G}(\tau_i, x)$ -game. Let α ’s first move be x_1 . By our assumption and the definition of $G_{ij}(\cdot, \cdot)$, there must exist some point $y_1 \in H$ such that $s(x_1) \cap U \cap g_j(1, y_1) \neq \emptyset$. Inductively, we can obtain two sequences $\{x_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$ in X such that for each $n \in \mathbb{N}$, $y_n \in H$ and

$$x_{n+1} \in U \cap g_j(n+1, y_{n+1}) \cap \left(\bigcap_{\substack{1 \leq j \leq n, \\ 1 \leq i_1 \leq \dots \leq i_j \leq n}} s(x_{i_1}, \dots, x_{i_j}) \right).$$

It follows that each subsequence of $\{x_n : n \in \mathbb{N}\}$ is an *s-sequence* in (X, τ_i) , and thus has an cluster point in (X, τ_i) . Since each point of X is a G_δ -point in (X, τ_i) , then $\{x_n : n \in \mathbb{N}\}$ must have a convergent subsequence, saying $\{x_{n_k} : k \in \mathbb{N}\}$, in (X, τ_i) . Suppose that $\{x_{n_k} : k \in \mathbb{N}\}$ is τ_i -convergent to some point $x_* \in \overline{U}^{\tau_i}$. Then, by condition (3) in Theorem 3.2, and the construction of $\{x_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$ in the above, $\{y_{n_k} : k \in \omega\}$ is also τ_i -convergent to x_* , and thus $x_* \in H$. It follows that $x_* \in \overline{U}^{\tau_i} \cap H$. We have derived a contradiction. Therefore, $x \notin \overline{G_{ij}(n, H)}^{\tau_i}$ for some $n \in \mathbb{N}$. We have verified that $H = \bigcap_{n \in \mathbb{N}} \overline{G_{ij}(n, H)}^{\tau_i}$ for all $H \in \mathcal{F}_i(X)$ and thus (X, τ_1, τ_2) is pairwise stratifiable. \square

5. QUASI-PSEUDO-METRIZABILITY OF TOPOLOGICAL ORDERED SPACES

In [15], Künzi and Mushaandja posed the following open problem (refer to [15, Problem 1]).

Problem 5.1 ([15]). *If (X, τ, \preccurlyeq) is a topological ordered C - and I -space such that the topology τ is metrizable, is the associated bitopological space (X, τ^b, τ^h) quasi-pseudo-metrizable?*

It was shown that if the topology τ is separable metrizable, then (X, τ^b, τ^h) is quasi-pseudo-metrizable. This result gives a partial affirmative answer to Problem 5.1 in the class of separable metrizable topological ordered C - and I -spaces. In this section, we provide another partial affirmative answer to this problem in the class of ball transitive and metrizable topological ordered C - and I -spaces. To this purpose, we first introduce the concept of ball transitivity.

An ordered metric space (X, ρ, \preccurlyeq) is said to be *ball transitive* [30] if there exists an $n \in \mathbb{N}$ such that whenever $x \preccurlyeq y$, then $B_\rho(x, \frac{\epsilon}{n}) \subseteq d(B_\rho(y, \epsilon))$ and $B_\rho(y, \frac{\epsilon}{n}) \subseteq i(B_\rho(x, \epsilon))$ hold for any $\epsilon > 0$. We call a metrizable topological ordered space (X, τ, \preccurlyeq) *ball transitive* provided that there is a metric ρ compatible with τ such that (X, ρ, \preccurlyeq) is ball transitive.

Remark 5.2. (i) Let X be the space of continuous real-valued functions on the interval $[0, 1]$. Let \preccurlyeq be the pointwise order on X and ρ be the metric defined by the sup-norm. It is well known that (X, ρ) is not separable, but it was shown in [30] that (X, ρ, \preccurlyeq) is ball transitive with $n = 1$.

(ii) Let A be the open first quadrant of \mathbb{R}^2 , i.e.,

$$A = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}.$$

Consider the following subset X of \mathbb{R}^2 ,

$$X = ([-1, 1] \times [-1, 1]) \setminus A,$$

equipped with the Euclidean metric ρ and the pointwise order \preccurlyeq . It is clear that (X, ρ, \preccurlyeq) is separable. However, it was shown in [30] that (X, ρ, \preccurlyeq) is not ball transitive.

Recall that a topological space (X, τ) is said to be a γ -space [10], provided that there is a g -function $g : \mathbb{N} \times X \rightarrow \tau$ such that if $x_n \in g(n, x)$ and $g_n \in g(n, x_n)$ for all $n \in \mathbb{N}$ then x is a cluster point of the sequence $\{y_n : n \in \mathbb{N}\}$. Herein, we call such a function g a γ -function for (X, τ) .

Theorem 5.3. *Let (X, τ, \preccurlyeq) be a metrizable topological ordered I -space. If (X, τ, \preccurlyeq) is ball transitive, then (X, τ^b) and (X, τ^h) are γ -spaces.*

Proof. Let ρ be a metric compatible with τ such that (X, ρ, \preccurlyeq) is ball transitive. Then, there is a $k \in \mathbb{N}$ such that whenever $x \preccurlyeq y$, $B_\rho(x, \frac{\epsilon}{k}) \subseteq d(B_\rho(y, \epsilon))$ and $B_\rho(y, \frac{\epsilon}{k}) \subseteq i(B_\rho(x, \epsilon))$ hold for any $\epsilon > 0$. For each $x \in X$, let $\mathcal{B}(x) = \{B_\rho(x, \frac{1}{2^n}) : n \in \mathbb{N}\}$. Since (X, τ, \preccurlyeq) is a metrizable I -space, then for any $x \in X$, $\mathcal{U}(x) = \{d(B_\rho(x, \frac{1}{2^n})) : n \in \mathbb{N}\}$ is a countable base of open neighbourhoods for τ^b at x and $\mathcal{V}(x) = \{i(B_\rho(x, \frac{1}{2^n})) : n \in \mathbb{N}\}$ is a countable base of open neighbourhoods for τ^h at x , respectively.

Next, define a g -function $g_1 : \mathbb{N} \times X \rightarrow \tau^b$ by letting $g_1(n, x) = d(B_\rho(x, \frac{1}{k4^n}))$ for each $x \in X$ and $n \in \mathbb{N}$. We verify that $g_1 : \mathbb{N} \times X \rightarrow \tau^b$ is a γ -function for (X, τ^b) . To this end, let $\{x_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$ be two sequences in X such that $x_n \in g_1(n, x)$ and $y_n \in g_1(n, x_n)$ for all $n \in \mathbb{N}$. Without loss of generality,

we may require $y_n \neq x$ for any $n \in \mathbb{N}$. Suppose that x is not a τ^b -cluster point of $\{y_n : n \in \mathbb{N}\}$. Then there exists an $m \in \mathbb{N}$ such that

$$d\left(B_\rho\left(x, \frac{1}{2^m}\right)\right) \subseteq X \setminus \overline{\{y_n : n \in \mathbb{N}\}}^{\tau^b}.$$

For each $n \geq m$, as $x_n \in g_1(n, x) = d(B_\rho(x, \frac{1}{k4^n}))$, there exists a $t_n \in B_\rho(x, \frac{1}{k4^n})$ such that $x_n \preccurlyeq t_n$. By the ball transitivity of (X, ρ, \preccurlyeq) , we have $B_\rho(x_n, \frac{1}{k4^m}) \subseteq d(B_\rho(t_n, \frac{1}{4^m}))$, which implies that

$$B_\rho(x_n, \frac{1}{k4^m}) \subseteq d\left(B_\rho(t_n, \frac{1}{4^m})\right) \subseteq d\left(B_\rho(x, \frac{1}{2^m})\right)$$

for all $n \geq m$. It follows that $y_n \in d(B_\rho(x, \frac{1}{2^m}))$ for all $n \geq m$. This is contradiction. Hence, x is not a τ^b -cluster point of $\{y_n : n \in \mathbb{N}\}$, which verifies that g_1 is a γ -function for (X, τ^b) .

Finally, define a g -function $g_2 : \mathbb{N} \times X \rightarrow \tau^b$ by letting $g_2(n, x) = i(B_\rho(x, \frac{1}{k4^n}))$ for each $x \in X$ and $n \in \mathbb{N}$. In the way similar to what we have done previously, we can prove that g_2 is a γ -function for (X, τ^b) . Therefore, both (X, τ^b) and (X, τ^b) are γ -spaces. \square

Lemma 5.4 ([20]). *A bisppace (X, τ_1, τ_2) is quasi-pseudo-metrizable if, and only if, (X, τ_1, τ_2) is pairwise stratifiable and (X, τ_i) is a γ -space for $i = 1, 2$.*

The following result provides a partial answer to Problem 5.1.

Theorem 5.5. *Let (X, τ, \preccurlyeq) be a topological ordered C - and I -space such that the topology τ is metrizable. If (X, τ, \preccurlyeq) is ball transitive, then (X, τ^b, τ^b) is quasi-pseudo-metrizable.*

Proof. First, by Theorem 5.3, both (X, τ^b) and (X, τ^b) are γ -spaces. Furthermore, by [15, Theorem 1], (X, τ^b, τ^b) is a pairwise stratifiable bisppace. Hence, it follows from [20, Theorem 4] that (X, τ^b, τ^b) is quasi-pseudo-metrizable. \square

Let (X, τ, \preccurlyeq) be a topological ordered C -space. It was shown in [15] that if τ is a stratifiable topology, then (X, τ^b, τ^b) is pairwise stratifiable. In addition, it was shown in [17] that if τ is a semi-stratifiable (resp. monotonically normal) topology, then (X, τ^b, τ^b) is pairwise semi-stratifiable (resp. monotonically normal). In the light of these results, we conclude this paper by posing the following open question.

Question 5.6. *Let (X, τ, \preccurlyeq) be a topological ordered C -space. If τ is a k -semi-stratifiable topology, must (X, τ^b, τ^b) be pairwise k -semi-stratifiable?*

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